QUANTUM HYPERPLANE SECTION PRINCIPLE FOR CONCAVEX DECOMPOSABLE VECTOR BUNDLES

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1. Introduction

The Lefschetz hyperplane section theorem says that there is an intimate relationship between the cohomology group of an ambient space and that of a smooth zero locus of a positive line bundle over the ambient space. Roughly speaking, it's quantum version says that there is an intimate relationship between quantum cohomology rings of the ambient space and that of a smooth zero locus of the decomposable spanned vector bundle [1, 2].

In paper [6], we proved the quantum analog when the ambient space is a generalized flag manifold and the decomposable vector bundle is convex. In it we claimed that the quantum analog can be generalized to the case when the bundle is concavex and decomposable. We explain the claim in this paper. As an application we reprove the multiple cover formula.

This work is originally motivated by the Lian-Liu-Yau paper [7]. A mirror theorem for concave bundle spaces over symplectic toric manifolds is established by Givental [5]. There is also a work of Elezi's paper [3] in the generalization of Givental's work for concavex decomposable vector bundle over projective spaces.

The result of the paper was announced in Bay Area Joint Symplectic Geometry Seminar in April, 1998 at Stanford University.

2. Notation

- Let X be a generalized flag manifold.
- Let V be a vector bundle decomposable to the direct sum of convex or concave line bundles L_j, $j = 1, ..., k$. A line bundle L is called convex if $H^1(\mathbb{P}^1, f^*(L_j)) = 0$ for any morphism $f : \mathbb{P}^1 \to X$. If $H^0(\mathbb{P}^1, f^*(L_j)) = 0$ for any morphism $f : \mathbb{P}^1 \to X$, L is called concave. We call V concavex and decomposable following [7].
- Let p_i, $i = 1, ..., l$, be the divisor classes of X such that $\bigoplus_{i=1}^{l} \mathbb{Z}_{\geq 0} p_i$ is the closed Kähler cone.

- Let q_i, $i = 1, ..., l$, denote indeterminants and $q^\beta := \prod_{i=1}^{l} q_i^{<p_i,\beta>}$, where β is an effective class of $H_2(X, \mathbb{Z})$. So, $q^\beta \in \mathbb{Q}[q_1, ..., q_l]$.
- Let $T' := \mathbb{C}^\times$, the complex torus, which acts on V fiberwise only using the scalar product of the vector spaces of fibers. Let T' act on the base space X trivially. So V is a T'-equivariant vector bundle.
- Denote by $H^*_{(T')}$ the rational field of the equivariant cohomology $H^*_{T'}(point, \mathbb{Q})$ of one point. Let $H^*_{(T')}(X) := H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*_{(T')}$
- Given an effective class $\beta \in H_2(X, \mathbb{Z})$,
$$\overline{M}_{0,n}(X, \beta)$$
denotes the moduli space of equivalence classes
$$[(C, f; x_1, ..., x_n)]$$
of n-marked stable maps of genus 0 and type β. An element of it can be represented by a holomorphic map f to X from a connected nodal curve C of arithmetic genus zero with $f_*([C]) = \beta \in H_2(X)$ and n-distinct ordered nonsingular points $x_i \in C$. The moduli space consists of stable ones. The space is a compact complex orbifold with dimension $\dim X + < c_1(X), \beta > +n - 3$ since T_X is generated by global sections.

3. Quantum Cohomology associated with (X, V)

Associated to the T'-equivariant bundle $V = \bigoplus_{j=1}^{k} L_j$ over X, we define a new quantum product on $H^*_{(T')}(X)$ following [4]. Modified Poincaré pairings based on V will be utilized.

Let A and B denote equivariant classes in $H^*_{(T')}(X)$. Define $<,>^V_0$, a nondegenerate inner product in $H^*_{(T')}(X)$ by

$$< A, B >^V_0 := \int_X ABE_{T'}(V),$$

where $E_{T'}(V) := \prod_i E_{T'}(L_i)$ and

$$E_{T'}(L) := \begin{cases} Euler_{T'}(L) & \text{if } L \text{ is convex} \\ Euler_{T'}(L)^{-1} & \text{if } L \text{ is concave.} \end{cases}$$

Here we use the T'-equivariant Euler classes $Euler_{T'}(L)$, so that the class $E_{T'}(V)$ is invertible over the coefficient ring $H^*_{(T')}$ and thus $<,>^V_0$ is indeed nondegenerate.

Introduce the induced T'-equivariant vector (orbi-)bundles

$$[L]_\beta := \begin{cases} R^0\pi_*(ev_{N+1})^*(L) & \text{if } L \text{ is convex} \\ R^1\pi_*(ev_{N+1})^*(L) & \text{if } L \text{ is concave} \end{cases},$$

where ev_{N+1} denotes the evaluation map at $(N+1)$-th marked points from $\overline{M}_{0,N+1}(X,\beta)$ to X and π denotes the forgetting-last-marked-point map from $\overline{M}_{0,N+1}(X,\beta)$ to $\overline{M}_{0,N}(X,\beta)$.

Let $A_i \in H^*_{(T')}(X)$, $i = 1, ..., N$. Define $<, ..., >^V_\beta$, N-correlators, by

$$< A_1, ..., A_N >^V_\beta = \int_{\overline{M}_{0,N}(X,\beta)} ev_1^*(A_1)...ev_N^*(A_N) Euler_{T'}(V_\beta)$$

where

$$V_\beta := \bigoplus_i [L_i]_\beta$$

In this definition if $\beta = 0$, then we assume $N \geq 3$. Here T' acts V_β fiberwise and

$$ev_i : \overline{M}_{0,N}(X,\beta) \to X, \quad i = 1, ..., N$$

are the evaluation maps at i-th marked points.

With these N- correlators and the nondegenerate pairing $<, >^L_0$, one can define a big/small quantum cohomology on $H^*_{(T')}(X)$ and also a quantum differential system, its fundamental solution, and so on. For instance, let A and B be in $H^*_{(T')}(X)$, then the small quantum product

$$A *_V B \in H^*_{(T')}(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[q]]$$

is defined by the requirement

$$< A *_V B, C >^V_0 = \sum_\beta q^\beta < A, B, C >^V_\beta,$$

for all $C \in H^*_{(T')}(X)$. So, $H^*_{(T')}(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, ..., q_l]]$ has the small quantum ring structure based on V and V_β.

Introduce formal parameters $t_1, ..., t_l$ and the relations $q_i = e^{t_i}$. The quantum differential system is a formal family of formal first order partial differential equations in $t_1, ..., t_l$ with $\hbar$ as a formal parameter:

$$p_i *_V f(t,q) = \hbar \frac{\partial}{\partial t_i} f(t,q)$$

where $f(t,q) \in H^*_{(T')}(X)[\hbar^{-1}][t_1, ..., t_l][[q_1, ..., q_l]]$. Here we treat $q_i \frac{\partial}{\partial q_i} = \frac{\partial}{\partial t_i}$ formally.

Let $(ev_N)_{*,V}$ *be defined as the adjoint of the pullback* ev_N^* *with respect to the* **new** *Poincaré pairings on the N-marked moduli spaces and* X. So, by the very definition of the pushforward,

$$\int_{\overline{M}_{0,N}(X,\beta)} A \cup ev_N^*(B) \cup Euler_{T'}(V_\beta) = \int_X (ev_N)_{*,V}(A) \cup B \cup E_{T'}(V)$$

for $A \in H^*_{(T')}(\overline{M}_{0,N}(X,\beta))$ and $B \in H^*_{(T')}(X)$.

Now we describe a fundamental solution to the quantum differential system. For any given $A \in H^*_{T'}(X)$,

$$f_A(t,q) := \sum_{\beta \neq 0} q^\beta (ev_2)_{*,V}(ev_1^*(A)\frac{\exp(ev_1^*(pt)/\hbar)}{\hbar - c}) + A\exp(pt/\hbar)$$

is a solution where $pt := \sum_i p_i t_i$ and c is the nonequivariant first Chern class of the universal cotangent line bundle at first marked points. To show it one may use WDVV, string and divisor equations [5] or use divisor equation and topological recursion relation [8].

4. The Givental correlator

Now define the so-called **Givental correlator**

$$J^V_\beta := ev_{*,V}(\frac{1}{\hbar(\hbar - c)}),$$

where $ev : \overline{M}_{0,1}(X,\beta) \to X$ is the evaluation map, and let $J^V_0 := 1$.

It is obtained from special components from solutions to the small quantum differential equations. That is,

$$< J^V_\beta, \exp(pt/\hbar)A >^V_0 = \int_{\overline{M}_{0,2}(X,\beta)} \frac{\exp(ev_1^*(pt)/\hbar)ev_1^*(A)}{\hbar - c} \cup ev_2^*(1) \cup Euler_{T'}(V_\beta)$$

if $\beta \neq 0$.

On the other hand, for $\beta \neq 0$,

$$< J^V_\beta, 1 >^V_0 = \frac{-2}{\hbar^3} \int_{\overline{M}_{0,0}(X,\beta)} Euler_{T'}(V_\beta) + o(\hbar^{-3}),$$

where the integral over the no-marked moduli space of $Euler(V_\beta)$ is sometimes very interesting. Some examples are as follows.

Example 1 If $X = \mathbb{P}^4$ and $V = \mathcal{O}(5)$, then the integration computes the degree of the virtual fundamental class of degree β for a smooth quintic. In turn it provides the integer numbers of "almost" rational curves of given homotopic types in the quintic defined by Ruan.

Example 2 If $X = \mathbb{P}^1$ and $V = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, the integral is the multiple covering contribution.

When V is the rank zero bundle (that is, there are no Euler things in correlators) we denote

$$J^X_\beta := ev_*(\frac{1}{\hbar(\hbar - c)}).$$

Example 3 When X is a projective space $\mathbb{P}^n$,

$$J^{\mathbb{P}^n} = 1 + \sum_{d=1}^{\infty} q^d \frac{1}{((p+\hbar)...(p+d\hbar))^{n+1}}$$

as in [4].

5. Quantum Hyperplane Section Principle

We want to compare J^V_β and J^X_β.

Define

$$H^L_\beta := \begin{cases} \prod_{m=1}^{<c_1(L),\beta>} (c_1^{T'}(L) + m\hbar) & \text{if } L \text{ is convex} \\ \\ \prod_{m=<c_1(L),\beta>+1}^{0} (c_1^{T'}(L) + m\hbar) & \text{if } L \text{ is concave.} \end{cases}$$

Let

$$H^V_\beta = \prod_i H^{L_i}_\beta$$

if $\beta \neq 0$ and $H^V_0 = 1$. **It will be called the correcting Euler class for V.** Here $c_1^{T'}(L)$ is the T'-equivariant first Chern class of L.

Define

$$J^V(q_1, ..., q_l) := \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta J^V_\beta$$

and

$$I^V(q_1, ..., q_l) := \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta J^X_\beta H^V_\beta .$$

The degree of q_i is uniquely defined by the requirement:

$$c_1(TX) - \sum_{\text{convex } L_i} c_1(L_i) + \sum_{\text{concave } L_i} c_1(L_i) = \sum_i (\deg q_i) p_i .$$

Theorem *Suppose each* $\deg q_i$ *is nonnegative. Then*

$$J^V = e^{f_0 + f_{-1}/\hbar + \sum p_i f_i/\hbar} I^V(q_1 e^{f_1}, ..., q_l e^{f_l})$$

for unique q-series f_i without constant terms where $\deg f_i = 0$ *for* $i = 0, ..., l$ *and* $\deg f_{-1} = 1$. *In particular, if* $I^V = 1 + O(\hbar^{-2})$, *then* $J^V = I^V$.

Proof. The proof in [6] for the convex case works for this general, concavex case, word for word.

Example of multiple cover formula In this case it is easy to see that I^V starts with $1 + O(\hbar^{-2})$ in the expansion in $\hbar^{-1}$. Therefore,

$J^V = I^V$ and thus $< J_d^V, 1 >_0^V = < I_d^V, 1 >_0^V = \int_{\mathbb{P}^1} \frac{1}{(p+d\hbar)^2} = -2\frac{1}{\hbar^3 d^3}$. We obtain the multiple cover formula which is first proven by Manin.

References

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